

22.101 Applied Nuclear Physics (Fall 2004)

Lecture 10 (10/18/04)

Nuclear Shell Model

References:

W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Chap.2.

P. Marmier and E. Sheldon, *Physics of Nuclei and Particles* (Academic Press, New York, 1969), vol. II, Chap.15.2.

Bernard L. Cohen, *Concepts of Nuclear Physics* (McGraw-Hill, New York, 1971).

There are similarities between the electronic structure of atoms and nuclear structure. Atomic electrons are arranged in orbits (energy states) subject to the laws of quantum mechanics. The distribution of electrons in these states follows the Pauli exclusion principle. Atomic electrons can be excited up to normally unoccupied states, or they can be removed completely from the atom. From such phenomena one can deduce the structure of atoms. In nuclei there are two groups of like particles, protons and neutrons. Each group is separately distributed over certain energy states subject also to the Pauli exclusion principle. Nuclei have excited states, and nucleons can be added to or removed from a nucleus.

Electrons and nucleons have intrinsic angular momenta called intrinsic spins. The total angular momentum of a system of interacting particles reflects the details of the forces between particles. For example, from the coupling of electron angular momentum in atoms we infer an interaction between the spin and the orbital motion of an electron in the field of the nucleus (the spin-orbit coupling). In nuclei there is also a coupling between the orbital motion of a nucleon and its intrinsic spin (but of different origin). In addition, nuclear forces between two nucleons depend strongly on the relative orientation of their spins.

The structure of nuclei is more complex than that of atoms. In an atom the nucleus provides a common center of attraction for all the electrons and inter-electronic forces generally play a small role. The predominant force (Coulomb) is well understood.

Nuclei, on the other hand, have no center of attraction; the nucleons are held together by their mutual interactions which are much more complicated than Coulomb interactions.

All atomic electrons are alike, whereas there are two kinds of nucleons. This allows a richer variety of structures. Notice that there are ~ 100 types of atoms, but more than 1000 different nuclides. Neither atomic nor nuclear structures can be understood without quantum mechanics.

Experimental Basis

There exists considerable experimental evidence pointing to the shell-like structure of nuclei, each nucleus being an assembly of nucleons. Each shell can be filled with a given number of nucleons of each kind. These numbers are called magic numbers; they are **2, 8, 20, 28, 50, 82, and 126**. (For the as yet undiscovered superheavy nuclei the magic numbers are expected to be $N = 184, 196, (272), 318$, and $Z = 114, (126), 164$ [Marmier and Sheldon, p. 1262].) Nuclei with magic number of neutrons or protons, or both, are found to be particularly stable, as can be seen from the following data.

- (i) Fig. 9.1 shows the abundance of stable isotones (same N) is particularly large for nuclei with magic neutron numbers.

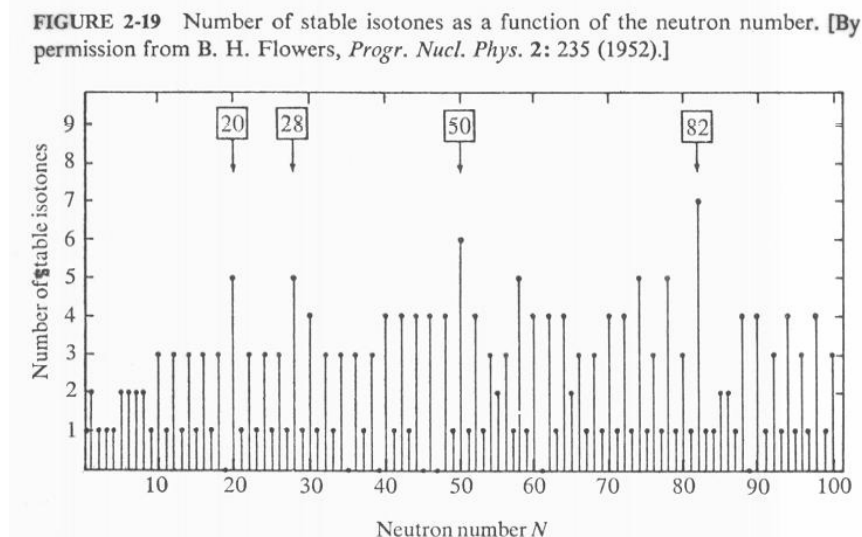


Fig. 9.1. Histogram of stable isotones showing nuclides with neutron numbers 20, 28, 50, and 82 are more abundant by 5 to 7 times than those with non-magic neutron numbers [from Meyerhof].

- (ii) Fig. 9.2 shows that the neutron separation energy S_n is particularly low for nuclei with one more neutron than the magic numbers, where

$$S_n = [M(A-1, Z) + M_n - M(A, Z)]c^2 \quad (9.1)$$

This means that nuclei with magic neutron numbers are more tightly bound.

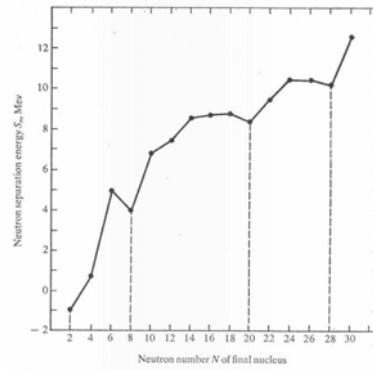


Fig. 9.2. Variation of neutron separation energy with neutron number of the final nucleus $M(A, Z)$ [from Meyerhof].

- (iii) The first excited states of even-even nuclei have higher than usual energies at the magic numbers, indicating that the magic nuclei are more tightly bound (see Fig. 9.3).

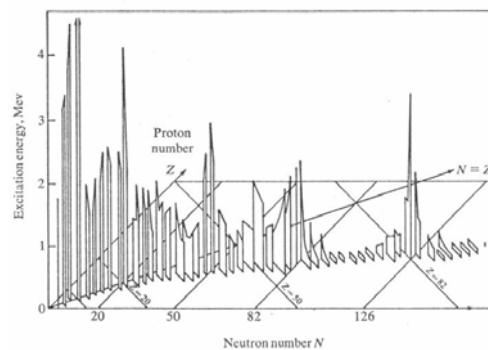


Fig. 9.3. First excited state energies of even-even nuclei [from Meyerhof].

- (iv) The neutron capture cross sections for magic nuclei are small, indicating a wider spacing of the energy levels just beyond a closed shell, as shown in Fig. 9.4.

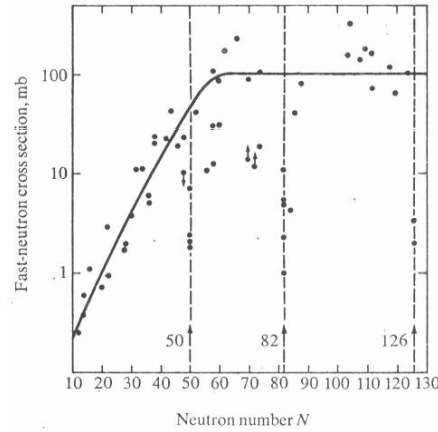


Fig. 9.4. Cross sections for capture at 1 Mev [from Meyerhof].

Simple Shell Model

The basic assumption of the shell model is that the effects of internuclear interactions can be represented by a single-particle potential. One might think that with very high density and strong forces, the nucleons would be colliding all the time and therefore cannot maintain a single-particle orbit. But, because of Pauli exclusion the nucleons are restricted to only a limited number of allowed orbits. A typical shell-model potential is

$$V(r) = -\frac{V_o}{1 + \exp[(r - R)/a]} \quad (9.1)$$

where typical values for the parameters are $V_o \sim 57$ Mev, $R \sim 1.25A^{1/3}$ F, $a \sim 0.65$ F. In addition one can consider corrections to the well depth arising from (i) symmetry energy from an unequal number of neutrons and protons, with a neutron being able to interact with a proton in more ways than n-n or p-p (therefore n-p force is stronger than n-n and p-p), and (ii) Coulomb repulsion. For a given spherically symmetric potential $V(r)$, one

can examine the bound-state energy levels that can be calculated from radial wave equation for a particular orbital angular momentum ℓ ,

$$-\frac{\hbar}{2m} \frac{d^2 u_\ell}{dr^2} + \left[\frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r) \right] u_\ell(r) = E u_\ell(r) \quad (9.2)$$

Fig. 9.5 shows the energy levels of the nucleons for an infinite spherical well and a harmonic oscillator potential, $V(r) = m\omega^2 r^2 / 2$. While no simple formulas can be given for the former, for the latter one has the expression

$$E_\nu = \hbar\omega(\nu + 3/2) = \hbar\omega(n_x + n_y + n_z + 3/2) \quad (9.3)$$

where $\nu = 0, 1, 2, \dots$, and $n_x, n_y, n_z = 0, 1, 2, \dots$ are quantum numbers. One should notice the degeneracy in the oscillator energy levels. The quantum number ν can be divided into *radial* quantum number n (1, 2, ...) and *orbital* quantum numbers ℓ (0, 1, ...) as shown in Fig. 9.5. One can see from these results that a central force potential is able to account for the first three magic numbers, 2, 8, 20, but not the remaining four, 28, 50, 82, 126. This situation does not change when more rounded potential forms are used. The implication is that something very fundamental about the single-particle interaction picture is missing in the description.

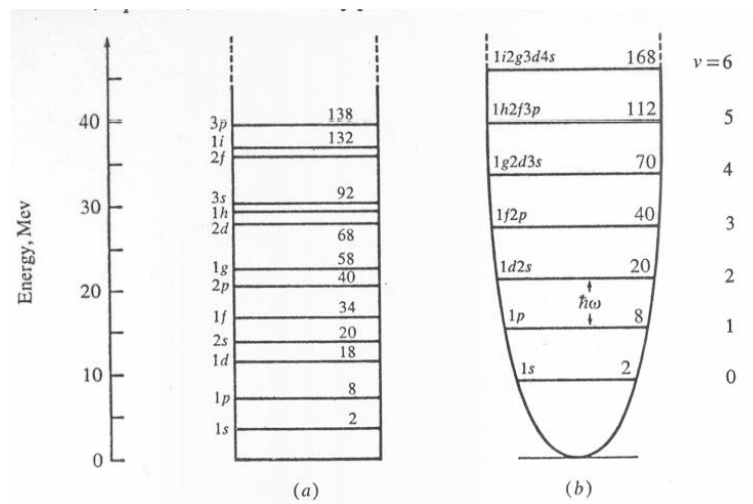


Fig. 9.5. Energy levels of nucleons in (a) infinite spherical well (range $R = 8F$) and (b) a parabolic potential well. In the spectroscopic notation (n, ℓ) , n refers to the number of times the orbital angular momentum state ℓ has appeared. Also shown at certain levels are the cumulative number of nucleons that can be put into all the levels up to the indicated level [from Meyerhof].

Shell Model with Spin-Orbit Coupling

It remains for M. G. Mayer and independently Haxel, Jensen, and Suess to show (1949) that an essential missing piece is an attractive interaction between the orbital angular momentum and the intrinsic spin angular momentum of the nucleon. To take into account this interaction we add a term to the Hamiltonian H ,

$$H = \frac{p^2}{2m} + V(r) + V_{so}(r) \underline{s} \cdot \underline{L} \quad (9.4)$$

where V_{so} is another central potential (known to be attractive). This modification means that the interaction is no longer spherically symmetric; the Hamiltonian now depends on the relative orientation of the spin and orbital angular momenta. It is beyond the scope of this class to go into the bound-state calculations for this Hamiltonian. In order to understand the meaning of the results of such calculations (eigenvalues and eigenfunctions) we need to digress somewhat to discuss the addition of two angular momentum operators.

The presence of the spin-orbit coupling term in (9.4) means that we will have a different set of eigenfunctions and eigenvalues for the new description. What are these new quantities relative to the eigenfunctions and eigenvalues we had for the problem without the spin-orbit coupling interaction? We first observe that in labeling the energy levels in Fig. 9.5 we had already taken into account the fact that the nucleon has an orbital angular momentum (it is in a state with a specified ℓ), and that it has an intrinsic spin of $\frac{1}{2}$ (in unit of \hbar). For this reason the number of nucleons that we can put into each level has been counted correctly. For example, in the $1s$ ground state one can put two

nucleons, for zero orbital angular momentum and two spin orientations (up and down). The student can verify that for a state of given ℓ , the number of nucleons that can go into that state is $2(2\ell + 1)$. This comes about because the eigenfunctions we are using to describe the system is a representation that *diagonalizes* the orbital angular momentum operator L^2 , its z-component, L_z , the intrinsic spin angular momentum operator S^2 , and its z-component S_z . Let us use the following notation to label these eigenfunctions (or representation),

$$|\ell, m_\ell, s, m_s\rangle \equiv Y_\ell^{m_\ell} \chi_s^{m_s} \quad (9.5)$$

where $Y_\ell^{m_\ell}$ is the spherical harmonic we encountered in Lec4, and we know it is the eigenfunction of the orbital angular momentum operator L^2 (it is also the eigenfunction of L_z). The function $\chi_s^{m_s}$ is the spin eigenfunction with the expected properties,

$$S^2 \chi_s^{m_s} = s(s+1)\hbar^2 \chi_s^{m_s}, \quad s=1/2 \quad (9.6)$$

$$S_z \chi_s^{m_s} = m_s \hbar \chi_s^{m_s}, \quad -s \leq m_s \leq s \quad (9.7)$$

The properties of $\chi_s^{m_s}$ with respect to operations by S^2 and S_z completely mirror the properties of $Y_\ell^{m_\ell}$ with respect to L^2 and L_z . Going back to our representation (9.5) we see that the eigenfunction is a “ket” with indices which are the good quantum numbers for the problem, namely, the orbital angular momentum and its projection (sometimes called the magnetic quantum number m , but here we use a subscript to denote that it goes with the orbital angular momentum), the spin (which has the fixed value of $1/2$) and its projection (which can be $+1/2$ or $-1/2$).

The representation given in (9.5) is no longer a good representation when the spin-orbit coupling term is added to the Hamiltonian. It turns out that the good representation is just a linear combination of the old representation. It is sufficient for our purpose to just know this, without going into the details of how to construct the linear

combination. To understand the properties of the new representation we now discuss angular momentum addition.

The two angular momenta we want to add are obviously the orbital angular momentum operator \underline{L} and the intrinsic spin angular momentum operator \underline{S} , since they are the only angular momentum operators in our problem. Why do we want to add them? The reason lies in (9.4). Notice that if we define the total angular momentum as

$$\underline{j} = \underline{S} + \underline{L} \quad (9.8)$$

we can then write

$$\underline{S} \cdot \underline{L} = (j^2 - S^2 - L^2) / 2 \quad (9.9)$$

so the problem of diagonalizing (9.4) is the same as diagonalizing j^2 , S^2 , and L^2 . This is then the basis for choosing our new representation. In analogy to (9.5) we will denote the new eigenfunctions by $|jm_j\ell s\rangle$, which has the properties

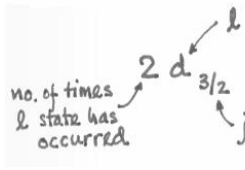
$$j^2 |jm_j\ell s\rangle = j(j+1)\hbar^2 |jm_j\ell s\rangle, \quad |\ell - s| \leq j \leq \ell + s \quad (9.10)$$

$$j_z |jm_j\ell s\rangle = m_j \hbar |jm_j\ell s\rangle, \quad -j \leq m_j \leq j \quad (9.11)$$

$$L^2 |jm_j\ell s\rangle = \ell(\ell+1)\hbar^2 |jm_j\ell s\rangle, \quad \ell = 0, 1, 2, \dots \quad (9.12)$$

$$S^2 |jm_j\ell s\rangle = s(s+1)\hbar^2 |jm_j\ell s\rangle, \quad s = 1/2 \quad (9.13)$$

In (9.10) we indicate the values that j can take for given ℓ and s ($=1/2$ in our discussion), the lower (upper) limit corresponds to when \underline{S} and \underline{L} are antiparallel (parallel) as shown in the sketch.



This is then the notation in which the shell-model energy levels are displayed in Fig. 9.6.

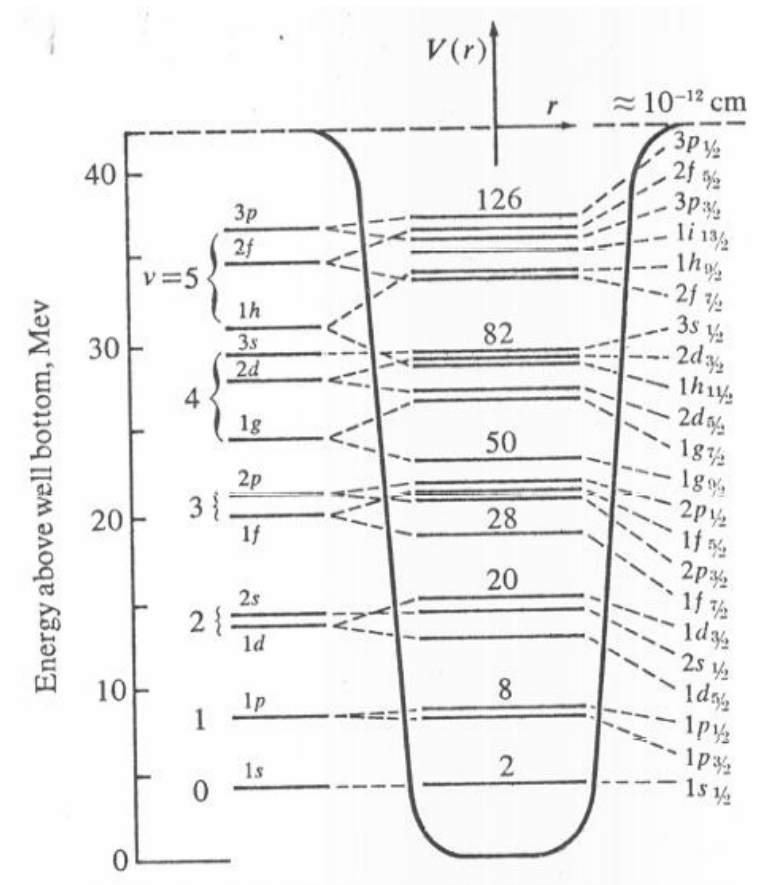


Fig. 9.6. Energy levels of nucleons in a smoothly varying potential well with a strong spin-orbit coupling term [from Meyerhof].

For a given (n, ℓ, j) level, the nucleon occupation number is $2j+1$. It would appear that having $2j+1$ identical nucleons occupying the same level would violate the Pauli exclusion principle. But this is not the case since each nucleon would have a distinct value of m_j (this is why there are $2j+1$ values of m_j for a given j).

We see in Fig. 9.6 the shell model with spin-orbit coupling gives a set of energy levels having breaks at the seven magic numbers. This is considered a major triumph of the model, for which Mayer and Jensen were awarded the Noble prize in physics. For our purpose we will use the results of the shell model to predict the ground-state spin and parity of nuclei. Before going into this discussion we leave the student with the following comments.

1. The shell model is most useful when applied to closed-shell or near closed-shell nuclei.
2. Away from closed-shell nuclei collective models taking into account the rotation and vibration of the nucleus are more appropriate.
3. Simple versions of the shell model do not take into account pairing forces, the effects of which are to make two like-nucleons combine to give zero orbital angular momentum.
4. Shell model does not treat distortion effects (deformed nuclei) due to the attraction between one or more outer nucleons and the closed-shell core. When the nuclear core is not spherical, it can exhibit “rotational” spectrum.

Prediction of Ground-State Spin and Parity

There are three general rules for using the shell model to predict the total angular momentum (spin) and parity of a nucleus in the ground state. These do not always work, especially away from the major shell breaks.

1. Angular momentum of odd-A nuclei is determined by the angular momentum of the last nucleon in the species (neutron or proton) that is odd.
2. Even-even nuclei have zero ground-state spin, because the net angular momentum associated with even N and even Z is zero, and even parity.

3. In odd-odd nuclei the last neutron couples to the last proton with their intrinsic spins in parallel orientation.

To illustrate how these rules work, we consider an example for each case. Consider the odd-A nuclide Be^9 which has 4 protons and 5 neutrons. Since the last nucleon is the fifth neutron, we see in Fig. 9.6 that this nucleon goes into the state $1p_{3/2}$ ($\ell=1, j=3/2$). Thus we would predict the spin and parity of this nuclide to be $3/2^-$. For an even-even nuclide we can take A^{36} , with 18 protons and neutrons, or Ca^{40} , with 20 protons and neutrons. For both cases we would predict spin and parity of 0^+ . For an odd-odd nuclide we take Cl^{38} , which has 17 protons and 21 neutrons. In Fig. 9.6 we see that the 17th proton goes into the state $1d_{3/2}$ ($\ell=2, j=3/2$), while the 21st neutron goes into the state $1f_{7/2}$ ($\ell=3, j=7/2$). From the ℓ and j values we know that for the last proton the orbital and spin angular momenta are pointing in opposite direction (because j is equal to $\ell - 1/2$). For the last neutron the two momenta are point in the same direction ($j = \ell + 1/2$). Now the rule tells us that the two spin momenta are parallel, therefore the orbital angular momentum of proton is pointing in the opposite direction from the orbital angular momentum of the neutron, with the latter in the same direction as the two spins. Adding up the four angular momenta, we have $+3+1/2+1/2-2 = 2$. Thus the total angular momentum (nuclear spin) is 2. What about the parity? The parity of the nuclide is the product of the two parities, one for the last proton and the other for the last neutron. Recall that the parity of a state is determined by the orbital angular momentum quantum number ℓ , $\pi = (-1)^\ell$. So with proton in a state with $\ell = 2$, its parity is even, while the neutron in a state with $\ell = 3$ has odd parity. The parity of the nucleus is therefore odd. Our prediction for Cl^{38} is then 2^- . The student can verify, using for example the Nuclide Chart, the foregoing predictions are in agreement with experiment.

Potential Wells for Neutrons and Protons

We summarize the qualitative features of the potential wells for neutrons and protons. If we exclude the Coulomb interaction for the moment, then the well for a proton is known to be deeper than that for a neutron. The reason is that in a given nucleus

usually there are more neutrons than protons, especially for the heavy nuclei, and the n-p interactions can occur in more ways than either the n-n or p-p interactions on account of the Pauli exclusion principle. The difference in well depth ΔV_s is called the symmetry energy; it is approximately given by

$$\Delta V_s = \pm 27 \frac{(N - Z)}{A} \text{ Mev} \quad (9.14)$$

where the (+) and (-) signs are for protons and neutrons respectively. If we now consider the Coulomb repulsion between protons, its effect is to raise the potential for a proton. In other words, the Coulomb effect is a positive contribution to the nuclear potential which is larger at the center than at the surface.

Combining the symmetry and the Coulomb effects we have a sketch of the potential for a neutron and a proton as indicated in Fig. 9.7. One can also estimate the

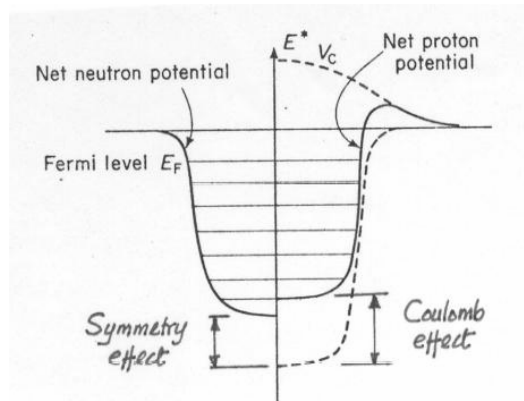


Fig. 9.7. Schematic showing the effects of symmetry and Coulomb interactions on the potential for a neutron and a proton [from Marmier and Sheldon].

Well depth in each case using the Fermi Gas model. One assumes the nucleons of a fixed kind behave like a fully degenerate gas of fermions (degeneracy here means that the states are filled continuously starting from the lowest energy state and there are no unoccupied states below the occupied ones), so that the number of states occupied is equal to the number of nucleons in the particular nucleus. This calculation is carried out

separately for neutrons and protons. The highest energy state that is occupied is called the *Fermi level*, and the magnitude of the difference between this state and the ground state is called the *Fermi energy* E_F . It turns out that E_F is proportional to $n^{2/3}$, where n is the number of nucleons of a given kind, therefore $E_F(\text{neutron}) > E_F(\text{proton})$. The sum of E_F and the separation energy of the last nucleon provides an estimate of the well depth. (The separation energy for a neutron or proton is about 8 Mev for many nuclei.) Based on these considerations one obtains the results shown in Fig. 9.8.

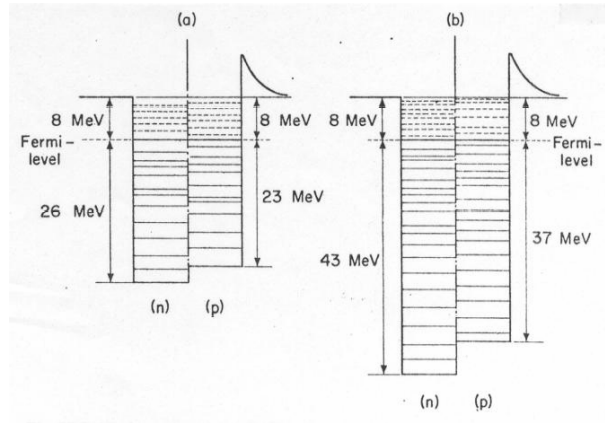


Fig. 9.8. Nuclear potential wells for neutrons and protons according to the Fermi-gas model, assuming the mean binding energy per nucleon to be 8 Mev, the mean relative nucleon admixture to be $N/A \sim 1/1.8$ $Z/A \sim 1/2.2$, and a range of 1.4 F (a) and 1.1 F (b) [from Marmier and Sheldon].

We have so far considered only a spherically symmetric nuclear potential well. We know there is in addition a centrifugal contribution of the form $\ell(\ell+1)\hbar^2 / 2mr^2$ and a spin-orbit contribution. As a result of the former the well becomes narrower and shallower for the higher orbital angular momentum states. Since the spin-orbit coupling is attractive, its effect depends on whether \underline{S} is parallel or anti-parallel to \underline{L} . The effects are illustrated in Figs. 9.9 and 9.10. Notice that for $\ell = 0$ both are absent.

We conclude this chapter with the remark that in addition to the bound states in the nuclear potential well there exist also virtual states (levels) which are not positive energy states in which the wave function is large within the potential well. This can

happen if the deBroglie wavelength is such that approximately standing waves are formed within the well. (Correspondingly, the reflection coefficient at the edge of the potential is large.) A virtual level is therefore not a bound state; on the other hand, there is a non-negligible probability that inside the nucleus a nucleon can be found in such a state. See Fig. 9.11.

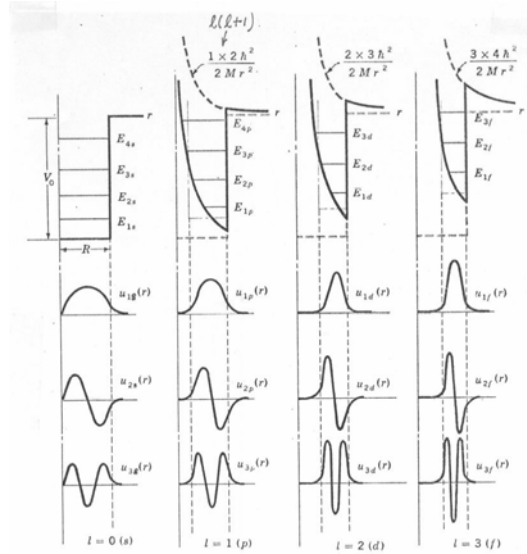


Fig. 9.9. Energy levels and wave functions for a square well for $\ell = 0, 1, 2$, and 3 [from Cohen]..

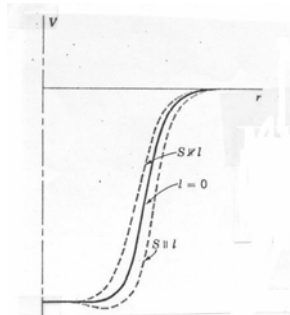


Fig. 9.10. The effect of spin-orbit interaction on the shell-model potential [from Cohen].

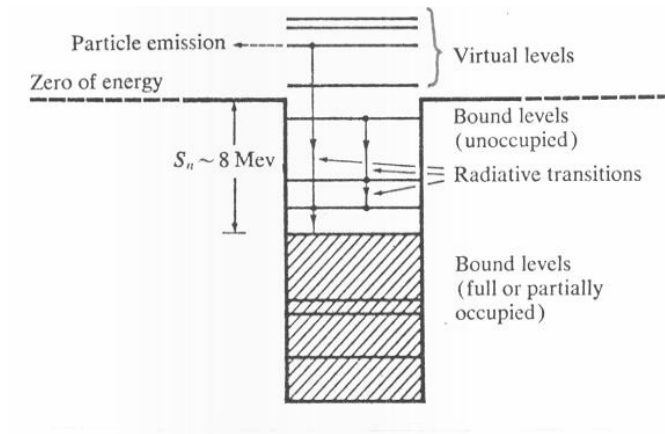


Fig. 9.11. Schematic representation of nuclear levels [from Meyerhof].

22.101 Applied Nuclear Physics (Fall 2004)

Lecture 11 (10/20/04)

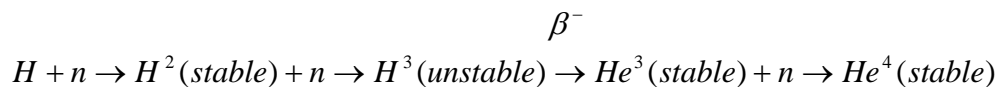
Nuclear Binding Energy and Stability

References:

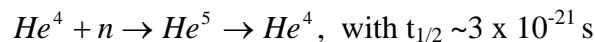
W. E. Meyerhof, *Elements of Nuclear Physics* (McGraw-Hill, New York, 1967), Chap.2.

The stability of nuclei is of great interest because unstable nuclei undergo transitions that result in the emission of particles and/or electromagnetic radiation (gammas). If the transition is spontaneous, it is called a radioactive decay. If the transition is induced by the bombardment of particles or radiation, then it is called a nuclear reaction.

The mass of a nucleus is the decisive factor governing its stability. Knowing the mass of a particular nucleus and those of the neighboring nuclei, one can tell whether or not the nucleus is stable. Yet the relation between mass and stability is complicated. Increasing the mass of a stable nucleus by adding a nucleon can make the resulting nucleus unstable, but this is not always true. Starting with the simplest nucleus, the proton, we can add one neutron after another. This would generate the series,

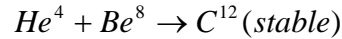
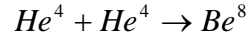


Because He^4 is a double-magic nucleus, it is particularly stable. If we continue to add a nucleon we find the resulting nucleus is unstable,



One may ask: With He^4 so stable how is it possible to build up the heavier elements starting with neutrons and protons? (This question arises in the study of the origin of

elements. See, for example, the Nobel lecture of A. Penzias in *Reviews of Modern Physics*, **51**, 425 (1979).) The answer is that the following reactions can occur



Although Be^8 is unstable, its lifetime of $\sim 3 \times 10^{-16}$ s is apparently long enough to enable the next reaction to proceed. Once C^{12} is formed, it can react with another He^4 to give O^{16} , and in this way the heavy elements can be formed.

Instead of the mass of a nucleus one can use the binding energy to express the same information. The binding energy concept is useful for discussing the calculation of nuclear masses and of energy released or absorbed in nuclear reactions.

Binding Energy and Separation Energy

We define the binding energy of a nucleus with mass $M(A,Z)$ as

$$B(A,Z) \equiv [ZM_H + NM_n - M(A,Z)]c^2 \quad (10.1)$$

where M_H is the hydrogen mass and $M(A,Z)$ is the atomic mass. Strictly speaking one should subtract out the binding energy of the electrons; however, the error in not doing so is quite small, so we will just ignore it. According to (10.1) the nuclear binding energy $B(A,Z)$ is the difference between the mass of the constituent nucleons, when they are far separated from each other, and the mass of the nucleus, when they are brought together to form the nucleus. Therefore, one can interpret $B(A,Z)$ as the *work required* to separate the nucleus into the individual nucleons (far separated from each other), or equivalently, as the *energy released* during the assembly of the nucleus from the constituents.

Taking the actual data on nuclear mass for various A and Z , one can calculate $B(A,Z)$ and plot the results in the form shown in Fig. 10.1.

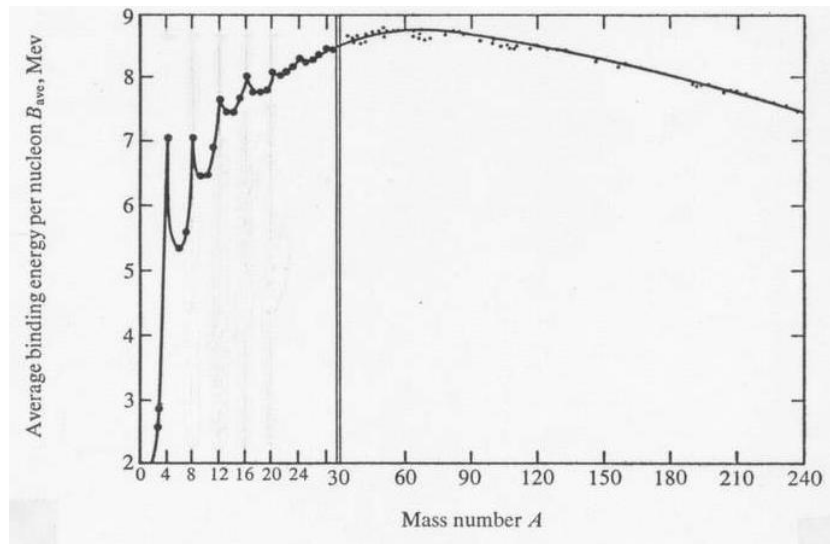


Fig. 10.1. Variation of average binding energy per nucleon with mass number for naturally occurring nuclides (and Be8). Note scale change at $A = 30$. [from Meyerhof]

The most striking feature of the B/A curve is the approximate constancy at ~ 8 Mev per nucleon, except for the very light nuclei. It is instructive to see what this behavior implies. If the binding energy of a pair of nucleons is a constant, say C , then for a nucleus with A nucleons, in which there are $A(A-1)/2$ distinct pairs of nucleons, the B/A would be $\sim C(A-1)/2$. Since this is not what one sees in Fig. 10.1, one can surmise that a given nucleon is not bound equally to all the other nucleons; in other words, nuclear forces, being short-ranged, extend over only a few neighbors. The constancy of B/A implies a saturation effect in nuclear forces, the interaction energy of a nucleon does not increase any further once it has acquired a certain number of neighbors. This number seems to be about 4 or 5.

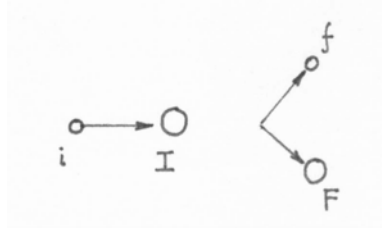
One can understand the initial rapid increase of B/A for the very light nuclei as the result of the competition between volume effects, which make B increase with A like A , and surface effects, which make B decrease (in the sense of a correction) with A like $A^{2/3}$. The latter should be less important as A becomes large, hence B/A increases (see the discussion of the semi-empirical mass formula in the next chapter). At the other end of the curve, the gradual decrease of B/A for $A > 100$ can be understood as the effect of

Coulomb repulsion which becomes more important as the number of protons in the nucleus increases.

As a quick application of the B/A curve we make a rough estimate of the energies release in fission and fusion reactions. Suppose we have symmetric fission of a nucleus with $A \sim 240$ producing two fragments, each $A/2$. The reaction gives a final state with B/A of about 8.5 Mev, which is about 1 Mev greater than the B/A of the initial state. Thus the energy released per fission reaction is about 240 Mev. (A more accurate estimate gives 200 Mev.) For fusion reaction we take $H^2 + H^2 \rightarrow He^4$. The B/A values of H^2 and He^4 are 1.1 and 7.1 Mev/nucleon respectively. The gain in B/A is 6 Mev/nucleon, so the energy released per fusion event is ~ 24 Mev.

Binding Energy in Nuclear Reactions

The binding energy concept is also applicable to a binary reaction where the initial state consists of a particle i incident upon a target nucleus I and the final state consists of an outgoing particle f and a residual nucleus F , as indicated in the sketch,



We write the reaction in the form

$$i + I \rightarrow f + F + Q \quad (10.2)$$

where Q is an energy called the ‘ Q -value of the reaction’. Corresponding to (10.2) we have the definition

$$Q \equiv [(M_i + M_I) - (M_f + M_F)]c^2 \quad (10.3)$$

where the masses are understood to be atomic masses. Every nuclear reaction has a characteristic Q-value; the reaction is called exothermic (endothermic) for $Q > 0$ (< 0) where energy is given off (absorbed). Thus an endothermic reaction cannot take place unless additional energy, called the threshold, is supplied. Often the source of this energy is the kinetic energy of the incident particle. One can also express Q in terms of the kinetic energies of the reactants and products of the reaction by invoking the conservation of *total* energy, which must hold for any reaction,

$$T_i + M_i c^2 + T_I + M_I c^2 \rightarrow T_f + M_f c^2 + T_F + M_F c^2 \quad (10.4)$$

Combining this with (10.3) gives

$$Q = T_f + T_F - (T_i + T_I) \quad (10.5)$$

Usually T_I is negligible compared to T_i because the target nucleus follows a Maxwellian distribution at the temperature of the target sample (typically room temperature), while for nuclear reactions the incident particle would have a kinetic energy in the Mev range. Since the rest masses can be expressed in terms of binding energies, another expression for Q is

$$Q = B(f) + B(F) - B(i) - B(I) \quad (10.6)$$

As an example, the nuclear medicine technique called boron neutron capture therapy (BNCT) is based on the reaction

$${}_5B^{10} + n \rightarrow {}_3Li^7 + He^4 + Q \quad (10.7)$$

In the case, $Q = B(Li) + B(\alpha) - B(B) = 39.245 + 28.296 - 64.750 = 2.791$ Mev. The reaction is exothermic, therefore it can be induced by a thermal neutron. In practice, the simplest way of calculating Q values is to use the rest masses of the reactants and products. For

many reactions of interest Q values are in the range 1 – 5 Mev. An important exception is fission, where $Q \sim 170 - 210$ Mev depending on what one considers to be the fission products. Notice that if one defines Q in terms of the kinetic energies, as in (10.5), it may appear that the value of Q would depend on whether one is in laboratory or center-of-mass coordinate system. This is illusory because the equivalent definition, (10.3), is clearly frame independent.

Separation Energy

Recall the definition of binding energy, (10.1), involves an initial state where all the nucleons are removed far from each other. One can define another binding energy where the initial state is one where only one nucleon is separated off. The energy required to separate particle a from a nucleus is called the separation energy S_a . This is also the energy released, or energy available for reaction, when particle a is captured. This concept is usually applied to a neutron, proton, deuteron, or α -particle. The energy balance in general is

$$S_a = [M_a(A', Z') + M(A - A', Z - Z') - M(A, Z)]c^2 \quad (10.8)$$

where particle a is treated as a ‘nucleus’ with atomic number Z' and mass number A' . For a neutron,

$$\begin{aligned} S_n &= [M_n + M(A - 1, Z) - M(A, Z)]c^2 \\ &= B(A, Z) - B(A - 1, Z) \end{aligned} \quad (10.9)$$

S_n is sometimes called the binding energy of the *last neutron*.

Clearly S_n will vary from one nucleus to another. In the range of A where B/A is roughly constant we can estimate from the B/A curve that $S_n \sim S_p \sim 8$ Mev. This is a rough figure, for the heavy nuclei S_n is more like 5 – 6 Mev. It turns out that when a nucleus $M(A-1, Z)$ absorbs a neutron, there is ~ 1 Mev (or more, can be up to 4 Mev) difference between the neutron absorbed being an even neutron or an odd neutron (see

Fig. 10.2). This difference is the reason that U^{235} can undergo fission with thermal neutrons, whereas U^{238} can fission only with fast neutrons ($E > 1$ Mev).

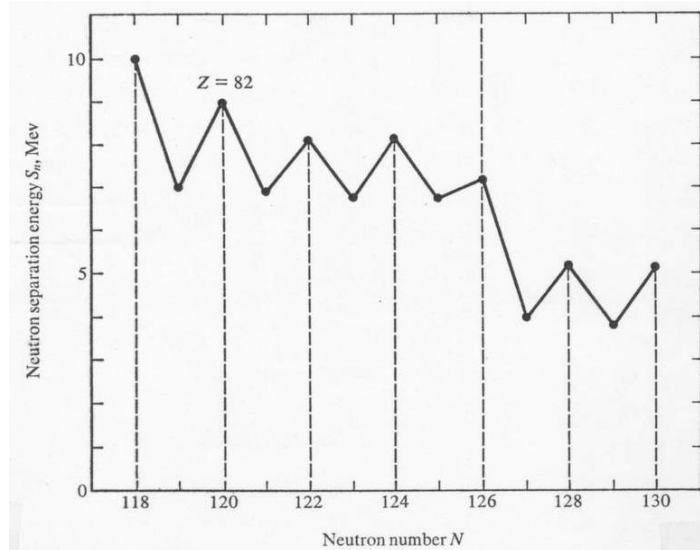


Fig. 10.2. Variation of the neutron separation energies of lead isotopes with neutron number of the absorbing nucleus. [from Meyerhof]

Generally speaking the following systematic behavior is observed in neutron and proton separation energies,

$$S_n(\text{even } N) > S_n(\text{odd } N) \quad \text{for a given } Z$$

$$S_p(\text{even } Z) > S_p(\text{odd } Z) \quad \text{for a given } N$$

This effect is attributed to the pairing property of nuclear forces – the existence of extra binding between pair of identical nucleons in the same state which have total angular momenta pointing in opposite directions. This is also the reason for the exceptional stability of the α -particle. Because of pairing the even-even (even Z , even N) nuclei are more stable than the even-odd and odd-even nuclei, which in turn are more stable than the odd-odd nuclei.

Abundance Systematics of Stable Nuclides

One can construct a stability chart by plotting the neutron number N versus the atomic number Z of all the stable nuclides. The results, shown in Fig. 10.3, show that $N \sim Z$ for low A , but $N > Z$ at high A .

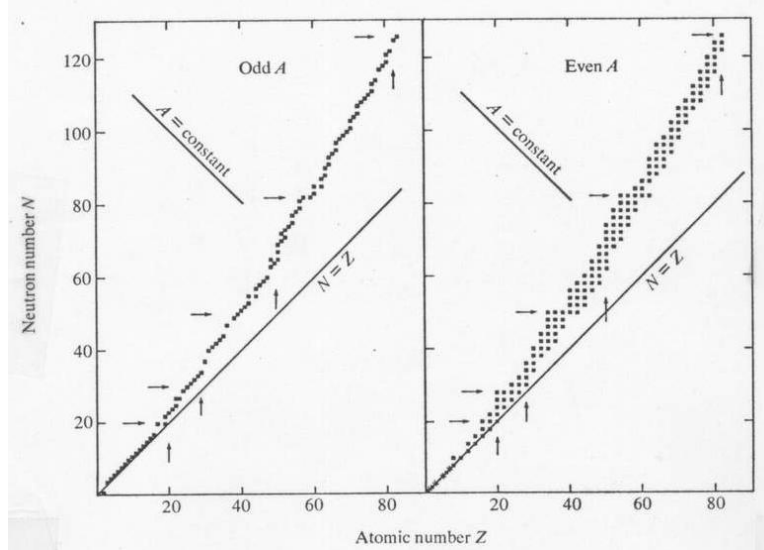


Fig. 10.3. Neutron and proton numbers of stable nuclides which are odd (left) and even isobars (right). Arrows indicate the magic numbers of 20, 28, 50, 82, and 126. Also shown are odd-odd isobars with $A = 2, 6, 10$, and 14. [from Meyerhof]

Again, one can readily understand that in heavy nuclei the Coulomb repulsion will favor a neutron-proton distribution with more neutrons than protons. It is a little more involved to explain why there should be an equal distribution for the light nuclides (see the following discussion on the semi-empirical mass formula). We will simply note that to have more neutrons than protons means that the nucleus has to be in a higher energy state, and is therefore less stable. This symmetry effect is most pronounced at low A and becomes less important at high A . In connection with Fig. 10.3 we note:

- (i) In the case of odd A , only one stable isobar exists, except $A = 113, 123$.
- (ii) In the case of even A , only even-even nuclides exist, except $A = 2, 6, 10, 14$.

Still another way to summarize the trend of stable nuclides is shown in the following table [from Meyerhof]

Table 2-2. SYSTEMATICS OF STABILITY TRENDS IN NUCLEI

<i>A</i>	<i>Z</i>	<i>N</i>	Type	Alternative designation	Number of stable + long-lived nuclides	Degree of stability	Usual number of stable isotopes per element
Even	Even	Even	e-e	Even mass, even <i>N</i>	166 + 11 = 177	Very pronounced	Several (2 and 3)
Odd	Even	Odd	e-o	Odd mass, odd <i>N</i>	55 + 3 = 58	Fair	1
Odd	Odd	Even	o-e	Odd mass, even <i>N</i>	51 + 3 = 54	Fair	1
Even	Odd	Odd	o-o	Even mass, odd <i>N</i>	6 + 4 = 10	Low	0
					278 + 21 = 299		